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# Comparing radiative and recoil corrections in neutron $\beta$-decay and inverse $\beta$-decay

U. Raha[1], F. Myhrer and K.Kubodera

Department of Physics and Astronomy, University of South Carolina, Columbia, SC 29208

**Abstract**

The inverse $\beta$-decay reaction, $\bar{\nu}_e p \to e^+ n$, for low-energy anti-neutrinos coming from nuclear reactors is of great current interest in connection with high-precision measurements of the neutrino mixing angle $\theta_{13}$. We have previously derived analytic expressions, up to next-to-leading order in heavy-baryon chiral perturbation theory, for the radiative corrections (RCs) and the nucleon-recoil corrections both for this reaction and for the related neutron $\beta$-decay process. We investigate here the numerical consequences of these analytic expressions. We show that the recoil corrections are small for neutron $\beta$-decay, but for inverse $\beta$-decay, the recoil corrections are comparable in size to the RCs for typical energies of reactor anti-neutrinos, and they have opposite signs. It turns out that the RCs and the recoil corrections exhibit very different dependences on the neutrino energy.

[1]Present address:
Department of Physics, Indian Institute of Technology Guwahati, Assam 781 039, India

Very recently, several experimental collaborations reported nonzero values of the neutrino mixing parameter, $\theta_{13}$ [1, 2, 3, 4]. Low-energy anti-neutrinos from nuclear reactors are well suited to determining $\theta_{13}$. The Double-Chooz [1], Daya Bay [2], and RENO [3] Collaborations have been carrying out high-precision measurements of $\theta_{13}$, by monitoring the inverse $\beta$-decay reaction

$$\bar{\nu}_e + p \to e^+ + n\,, \tag{1}$$

for the reactor-generated $\bar{\nu}_e$'s. The accurate extraction of $\theta_{13}$ from a measured positron yield in the reaction Eq.(1) requires a precise knowledge of radiative corrections (RCs) and nucleon-recoil corrections. An important issue in this connection is to what extent one can exploit the experimental data on neutron $\beta$-decay

$$n \to p + e^- + \bar{\nu}_e\,, \tag{2}$$

to have a better control of the RCs and recoil corrections to the inverse $\beta$-decay reaction.

In earlier works [5, 6, 7], the relevant RCs were evaluated to order-$\alpha$ in the theoretical framework developed by Sirlin and Marciano, to be referred to as the S-M approach; see, *e.g.*, Refs. [8, 9]. Although the estimates based on the S-M approach are believed to be reliable to the level of accuracy quoted in the literature, it is not totally excluded that these estimates may involve some degree of model dependence. Meanwhile, the nucleon-recoil corrections for the inverse $\beta$-decay process were evaluated based on the $1/m_N$-expansion of the nucleon weak-interaction form factors, where $m_N$ is the nucleon mass; see e.g., Refs. [6, 10].

In Refs. [11, 12], we proposed to use an effective field theory (EFT) approach to derive the RCs and nucleon-recoil corrections for the processes, Eq.(1) and Eq.(2). Analytic expressions for both the RCs and the recoil corrections to next-to-leading order (NLO) were presented for neutron $\beta$-decay [11], and for inverse $\beta$-decay [12]. Our analytic RC expressions are consistent with those obtained in the S-M approach by Fukugita and Kubota [5], and by Sirlin [13]; see also Refs. [6, 7] for earlier works. As for the recoil corrections, our analytic results for inverse $\beta$-decay presented in Ref. [12] were found to be consistent with Ref. [10]. It was noted, however, that the recoil corrections (in particular, the one arising from the weak-magnetism term) have very different analytic expressions for neutron $\beta$-decay and the inverse $\beta$-decay reaction.

In Refs. [11, 12], we did not go into detailed examinations of the numerical consequences of our analytic results. The purpose of the present note is to supply such examinations, which we hope will shed further light on the practical significance of

the RCs and the recoil corrections. In this paper, after a brief explanation of the theoretical framework used, we present numerical results for the RCs and recoil corrections calculated up to NLO for representative observables (see below) for neutron $\beta$-decay and inverse $\beta$-decay.

Let $\bar{Q}$ represent the typical four-momentum involved in neuron $\beta$-decay, or inverse $\beta$-decay. For neutron $\beta$-decay, because of its low $Q$-value ($Q_\beta = 0.782$ MeV), $\bar{Q}$ is very small compared with the nucleon mass; in fact, it is very small even compared with the pion mass. The same is true with inverse $\beta$-decay (with $Q_{inv-\beta} = m_p - m_n - m_e \simeq -1.8$ MeV), so long as we only consider cases where incident anti-neutrinos are those coming from nuclear reactors. Therefore, the two processes under consideration are naturally described by effective field theory (EFT). In particular, the well-established chiral perturbation theory ($\chi$PT), see e.g., Refs. [14, 15, 16], provides an ideal framework to carry out a systematic calculation of the electroweak RCs in a *model-independent* and *gauge-invariant* manner. $\chi$PT treats all strong-interaction and electroweak corrections within a *unified* power-counting scheme. This EFT framework admits a perturbative expansion of relevant Feynman amplitudes in terms of two expansion parameters: (i) the $\chi$PT expansion parameter, $\bar{Q}/\Lambda_\chi$, where $\Lambda_\chi \simeq 4\pi f_\pi \approx 1$ GeV ($f_\pi =$ pion-decay constant) is the chiral scale; (ii) the usual QED expansion parameter, $\alpha/(2\pi)$, which governs the contributions from the electroweak RCs. We adopt here heavy-baryon chiral perturbartion theory (HB$\chi$PT); see e.g., Ref. [14]. In this scheme an additional expansion parameter, $\bar{Q}/m_N$, governs the contributions from nucleon-recoil corrections. Since $\Lambda_\chi \simeq m_N \approx 1$ GeV, the $\bar{Q}/\Lambda_\chi$ and $\bar{Q}/m_N$ can be considered as one expansion parameter. Thus, the $m_{\rm N}^{-1}$-expansion is a natural part of our counting scheme, which dictates how to systematically incorporate the recoil corrections. In EFT the short-distance physics, which is not probed in low-energy processes, is subsumed into so-called *low-energy constants* (LECs). Although in principle the LECs can be evaluated from the fundamental theory, i.e., QCD, but a more pragmatic approach is to determine the LECs by fitting experimental observables. Once the LECs are known, we can make predictions for other measurable quantities. Our concern here is to carry out a HB$\chi$PT-based calculation of the RCs and the recoil corrections up to NLO, *viz.,* first order in $\alpha/2\pi$, $\bar{Q}/\Lambda_\chi$, and $\bar{Q}/m_N$.

The effective lagrangian relevant to our calculation includes the relativistic leptonic weak interaction current and the heavy-baryon chiral lagrangian. To NLO, it is given by

$$\mathcal{L}_{eff} \;=\; \mathcal{L}_{QED} + \mathcal{L}_{NN} + \mathcal{L}_{NN\psi\psi} \;, \tag{3}$$

where

$$
\begin{aligned}
\mathcal{L}_{QED} &= -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2\xi_A}(\partial\cdot A)^2 + \left(1+\frac{\alpha}{4\pi}e_1\right)\bar{\psi}_e(i\gamma\cdot D)\psi_e + m_e\bar{\psi}_e\psi_e + \bar{\psi}_\nu i\gamma\cdot\partial\psi_\nu, \quad (4)\\
\mathcal{L}_{NN} &= \bar{N}\left[1+\frac{\alpha}{8\pi}e_2(1+\tau_3)\right](iv\cdot D)N\,, \quad (5)\\
\mathcal{L}_{NN\psi\psi} &= -\left(\frac{G_F V_{ud}}{\sqrt{2}}\right)\left[\bar{\psi}_e\gamma_\mu(1-\gamma_5)\psi_\nu\right]\Big\{\bar{N}\tau^+\Big[\left(1+\frac{\alpha}{4\pi}e_{\rm V}\right)v^\mu - 2g_A\left(1+\frac{\alpha}{4\pi}e_{\rm A}\right)S^\mu\Big]N\\
&\quad + \frac{1}{2m_N}\bar{N}\tau^+\Big[i(v^\mu v^\nu - g^{\mu\nu})(\overleftarrow{\partial}-\overrightarrow{\partial})_\nu - 2i\mu_{\rm V}\Big[S^\mu, S\cdot(\overleftarrow{\partial}+\overrightarrow{\partial})\Big] - 2ig_{\rm A}v^\mu S\cdot(\overleftarrow{\partial}-\overrightarrow{\partial})\Big]N\Big\}\,. \quad (6)
\end{aligned}
$$

$\mathcal{L}_{QED}$ in Eq.(4) is the usual QED lagrangian, where $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ is the electromegnetic field tensor, and $D_\mu=\partial_\mu+ieA_\mu$ is the covariant derivative. The Feynman gauge is used to fix the gauge parameter, i.e., $\xi_A=1$. The $\mathcal{L}_{NN}$ is the part of the HB$\chi$PT lagrangian that includes the photon-nucleon interactions, while $\mathcal{L}_{NN\psi\psi}$ is the part that represents the low-energy LO and NLO current-current weak interactions. Eq.(6) also contains the explicit forms of NLO nucleon-recoil terms dictated by HB$\chi$PT. Here $g_A=1.267$ is the axial coupling constant, while $v_\mu$ is the nucleon velocity four-vector, and $S^\mu$ is the nucleon spin; they satisfy $v\cdot S=0$. It is convenient to choose $v^\mu=(1,\vec{0})$ and $S^\mu=(0,\vec{\sigma}/2)$. In the NLO part of the lagrangian, $\mu_{\rm V}=\mu_p-\mu_n=4.706$ is the nucleon isovector magnetic moment. The low-energy constants (LECs), $e_1$, $e_2$, $e_{\rm V}$ and $e_{\rm A}$, incorporate the short-range radiative physics that is not probed in a low-energy process. The Fermi coupling constant, $G_F=1.166\times10^{-5}$ GeV$^{-2}$, is determined from muon decay, and the CKM matrix element, $|V_{ud}|=0.97418\pm0.00027$, is given by the PDG [17].

We next briefly recapitulate the EFT-based calculations of the NLO radiative and recoil corrections described in Ref.[11] and Ref.[12] for neutron $\beta$-decay and inverse $\beta$-decay, respectively. We shall focus on experimental situations where none of the particle spins are monitored by the detectors. Furthermore, for the sake of definiteness, we concentrate on the differential decay rate for the neutron $\beta$-decay process, $n(p_n)\to p(p_p)+e^-(p)+\bar{\nu}_e(p_\nu)$, and the differential cross section for the inverse $\beta$-decay reation, $\bar{\nu}_e(p_\nu)+p(p_p)\to e^+(p)+n(p_n)$. Here the four-momenta of particles are indicated in the parentheses; for the four-momenta, $p$ and $p_\nu$, we shall also use quantities defined by $p=(E,\boldsymbol{p})$ and $p_\nu=(E_\nu,\boldsymbol{p}_\nu)$.

For neutron $\beta$-decay, from Ref. [11], we have the differential decay rate

$$\frac{\mathrm{d}\Gamma_\beta}{\mathrm{d}E\ \mathrm{d}(\cos\theta_e)} = \left(\frac{G_F V_{ud}}{\sqrt{2}}\right)^2 g(E)\left[(1{+}3g_A^2)\,\mathcal{K}_1(\beta) + (1{-}g_A^2)\,\mathcal{K}_2(\beta)\beta\cos\theta_e\right], \quad (7)$$

where $\mathcal{K}_i(\beta)$'s $(i=1,\ 2)$ contain the radiative and recoil corrections, to be described below. Meanwhile, according to Ref. [12], the differential cross-section for inverse $\beta$-decay is given by

$$\frac{\mathrm{d}\sigma_{inv-\beta}}{\mathrm{d}(\cos\theta_e)} = \left(\frac{G_F V_{ud}}{\sqrt{2}}\right)^2 f(E)\left[(1{+}3g_A^2)\,\mathcal{G}_1(\beta) + (1{-}g_A^2)\,\mathcal{G}_2(\beta)\beta\cos\theta_e\right], \quad (8)$$

where $\mathcal{G}_i(\beta)$'s $(i=1,2)$ contain the radiative and recoil corrections, see below. In Eqs.(7) and (8), $\beta{=}|\boldsymbol{p}|/E{=}\sqrt{E^2-m_e^2}/E$ is the velocity of the outgoing electron/positron and $\cos(\theta_e){=}\hat{\boldsymbol{p}}_\nu\cdot\hat{\boldsymbol{p}}$. The phase-space factor, $g(E)$, that includes the $\mathcal{O}(m_N^{-1})$ corrections is given by [11]

$$g(E) = \frac{F(Z,E)\,E^2\beta\,(E^{max}{-}E)^2}{2\pi^3}\left[1+\frac{1}{m_N}(3E-E^{max}-3\beta E\cos\theta_e)+\mathcal{O}(m_N^{-2})\right], \quad (9)$$

where $E^{max}=(m_n^2-m_p^2+m_e^2)/2m_n = E_\nu+E+\mathcal{O}(m_N^{-1})$ is the maximum or "end-point" energy of the emitted electron in neutron $\beta$-decay, and $F(Z,E)\approx 1{+}\alpha\pi/\beta$ $(Z{=}1)$ is the usual Fermi-Coulomb function. The phase-space factor, $f(E)$, in Eq.(8) was obtained in Ref. [12] as[2]

$$f(E) = \frac{E^2\beta}{\pi}\left[1-\frac{1}{m_N}\left(E-\frac{E_\nu}{\beta}\cos\theta_e\right)+\mathcal{O}(m_N^{-2})\right]. \quad (10)$$

The above velocity-dependent functions, $\mathcal{K}_i(\beta)$ and $\mathcal{G}_i(\beta)$ $(i=1,\ 2)$, contain terms coming from the RCs and those from nucleon-recoil corrections,

$$\mathcal{K}_i(\beta) = 1+\frac{\alpha}{2\pi}\,\mathcal{K}_i^{rad}(\beta)+\frac{1}{m_{\rm N}}\,\mathcal{K}_i^{recoil}(\beta) \quad (11)$$

$$\mathcal{G}_i(\beta) = 1+\frac{\alpha}{2\pi}\,\mathcal{G}_i^{rad}(\beta)+\frac{1}{m_{\rm N}}\,\mathcal{G}_i^{recoil}(\beta)\,. \quad (12)$$

[2] Note that for inverse $\beta$-decay, the positron energy, $E$, and velocity, $\beta$, contain terms of $\mathcal{O}(m_N^{-1})$, e.g., $E=\tilde{E}(1-\mathcal{O}(m_N^{-1}))$, where $\tilde{E}=E_\nu-(m_n-m_p)$; see the discussion leading to Eq. (14) in Ref. [12].

As for the recoil corrections, we note that, since the kinematical $1/m_N$ corrections are already included in the phase-space factors, $g(E)$ and $f(E)$, the recoil corrections in Eqs.(11),(12) are dynamical ones coming from the $\mathcal{O}(m_N^{-1})$ terms in Eq.(6).

The HB$\chi$PT calculation, up to NLO, of the RCs was given in Ref. [11] for neutron $\beta$-decay, and in Ref. [12] for inverse $\beta$-decay. Here we simply quote the results, referring the reader to Refs. [11] and [12] for the details of their derivation.[3] The results are given as:

$$1+\frac{\alpha}{2\pi}\,\mathcal{K}_1^{rad}(\beta) = \left[1+\frac{\alpha}{2\pi}\,\tilde{e}_V^R(\mu^2)\right]\left[1+\frac{\alpha}{2\pi}\left(\delta_\alpha^{(1)}(\beta)-\frac{5}{4}\right)\right] \quad (13)$$

$$1+\frac{\alpha}{2\pi}\,\mathcal{K}_2^{rad}(\beta) = \left[1+\frac{\alpha}{2\pi}\tilde{e}_V^R(\mu^2)\right]\left[1+\frac{\alpha}{2\pi}\left(\delta_\alpha^{(1)}(\beta)+\delta_\alpha^{(2)}(\beta)-\frac{5}{4}\right)\right], \quad (14)$$

$$1+\frac{\alpha}{2\pi}\,\mathcal{G}_1^{rad}(\beta) = \left[1+\frac{\alpha}{2\pi}\,\tilde{e}_V^R(\mu^2)\right]\left[1+\frac{\alpha}{2\pi}\delta_{out}(\beta)\right] \quad (15)$$

$$1+\frac{\alpha}{2\pi}\,\mathcal{G}_2^{rad}(\beta) = \left[1+\frac{\alpha}{2\pi}\tilde{e}_V^R(\mu^2)\right]\left[1+\frac{\alpha}{2\pi}\tilde{\delta}_{out}(\beta)\right]. \quad (16)$$

The so-called "inner" RCs, which are independent of $\beta$, are encoded in the LEC, $\tilde{e}_V^R(\mu^2)$. The "outer" RCs, $\delta_\alpha^{(1)}(\beta)$, $\delta_\alpha^{(2)}(\beta)$, $\delta_{out}(\beta)$ and $\tilde{\delta}_{out}(\beta)$, constitute the well-known, model-independent, long-distance QED corrections that do not contain any hadronic effects. Analytic expressions for these outer corrections can be found in, e.g., Refs. [11, 12]. Note that not only the ultraviolet-divergent and scale dependent terms are subsumed in the LEC, $\tilde{e}_V^R(\mu^2)$, but also all the infrared-divergent terms of $\mathcal{O}(\alpha)$ are simultaneously canceled, leading to finite final results.

As for the recoil corrections, $\mathcal{K}_i^{recoil}(\beta)$'s $(i=1,2)$ appearing in Eq.(11) (pertaining to neutron $\beta$-decay), and $\mathcal{G}_i^{recoil}(\beta)$'s $(i=1,2)$ appearing in Eq.(12) (pertaining to inverse $\beta$-decay rate) are given by (see Refs. [11, 12]),

$$\begin{aligned}
\mathcal{K}_1^{recoil}(\beta) &= \beta^2 E\left(\frac{1+2g_{\rm A}\mu_{\rm V}+g_{\rm A}^2}{1+3g_{\rm A}^2}\right)+E_\nu\left(\frac{1-2g_{\rm A}\mu_{\rm V}+g_{\rm A}^2}{1+3g_{\rm A}^2}\right)\\
\mathcal{K}_2^{recoil}(\beta) &= E\left(\frac{1-2g_{\rm A}\mu_{\rm V}+g_{\rm A}^2}{1-g_{\rm A}^2}\right)+E_\nu\left(\frac{1+2g_{\rm A}\mu_{\rm V}+g_{\rm A}^2}{1-g_{\rm A}^2}\right).
\end{aligned} \quad (17)$$

$$\begin{aligned}
\mathcal{G}_1^{recoil}(\beta) &= \beta^2 E\left(\frac{1-2g_{\rm A}\mu_{\rm V}+g_{\rm A}^2}{1+3g_{\rm A}^2}\right)-E_\nu\left(\frac{1+2g_{\rm A}\mu_{\rm V}+g_{\rm A}^2}{1+3g_{\rm A}^2}\right)\\
\mathcal{G}_2^{recoil}(\beta) &= E\left(\frac{1+2g_{\rm A}\mu_{\rm V}+g_{\rm A}^2}{1-g_{\rm A}^2}\right)-E_\nu\left(\frac{1-2g_{\rm A}\mu_{\rm V}+g_{\rm A}^2}{1-g_{\rm A}^2}\right).
\end{aligned} \quad (18)$$

[3] It is to be remarked that the HB$\chi$PT results are consistent with those obtained in the the S-M approach [5, 6, 7, 8].

A noteworthy point in the above expressions is that the terms involving $2g_A\mu_V \simeq 12$ arising from the "weak-magnetism" interactions in Eq.(6) constitute the most dominant recoil corrections.

The above formulae allow us to take a formally consistent account (up to NLO in HB$\chi$PT) of the RCs and recoil corrections for both neutron $\beta$-decay and the inverse $\beta$ process. Since these two processes involve the same LEC, $\tilde{e}_V^R(\mu^2)$, as the only unknown parameter, it is in principle possible to use neutron $\beta$-decay data to fix $\tilde{e}_V^R(\mu^2)$, and make model-independent predictions for the inverse $\beta$-decay cross sections. This statement is true to the extent that the other quantities which go into the above formulae and which are usually treated as "known" quantities are indeed known with high enough precision. To what degree the existing uncertainties in the values of $g_A$ and the neutron lifetime affect the outcome of such an analysis is a subject that deserves a careful study. Relegating this investigation to a future study, we choose here to use an estimate for $\tilde{e}_V^R(\mu^2)$ deduced in Ref. [11] by comparing the results for neutron $\beta$-decay obtained in HB$\chi$PT with those obtained in the S-M approach [8, 9]. This comparison yields

$$\tilde{e}_V^R(\mu^2{=}m_{\rm N}^2) \approx 4\ln\left(\frac{m_Z}{m_p}\right) + \ln\left(\frac{m_p}{m_A}\right) + 2C + A_g = 19.5 \pm 0.7\,, \tag{19}$$

where the following input has been used. From a "liberal" choice of the $A_1$-resonance mass, one has $m_A \approx 1.2 \pm 0.6$ MeV; the electroweak "loop"-function, $C$, is estimated to be $C = 0.829(1 \pm 0.1)$, while a perturbative QCD calculation gives $A_g = -0.34$ (see, e.g., Ref. [18]). Thus the size of the RC involving the LEC in Eqs. (13)$\sim$(16) is estimated to be $(\alpha/2\pi)\,\tilde{e}_V^R(m_{\rm N}^2) \simeq 0.023$. This short-distance (inner) contribution to $\tilde{e}_V^R(m_{\rm N}^2)$ is dominated by the well-known box diagrams involving $Z$-boson exchanges [8, 9]. The numerical results reported in what follows have been obtained with the use of $\tilde{e}_V^R(\mu^2{=}m_{\rm N}^2) = 19.5 \pm 0.7$,

To illustrate interplay between the RCs and recoil corrections, and to exemplify the differences between the recoil corrections for $\beta$-decay and inverse $\beta$-decay, we concentrate on the angle-integrated observables, *viz.,* the differential decay rate, $\mathrm{d}\Gamma_\beta/\mathrm{d}E$, for $\beta$-decay, and the total cross section, $\sigma_{inv-\beta}$, for the inverse $\beta$-decay reaction. This means that we are concerned with $\mathcal{K}_1$ and $\mathcal{G}_1$ in Eqs.(7), (8), or equivalently, with $\mathcal{K}_1^{rad}$, $\mathcal{G}_1^{rad}$, $\mathcal{K}_1^{recoil}$, and $\mathcal{G}_1^{recoil}$ in Eqs.(11), (12).

In Fig.1 we plot $\frac{\alpha}{2\pi}\mathcal{K}_1^{rad}$ for neutron $\beta$-decay as a function of the anti-neutrino energy, $E_\nu$ $(0 \le E_\nu \le 0.78\,\mathrm{MeV})$. The pair of RC curves in the figure indicates an error-band associated with the above-mentioned uncertainty in $\tilde{e}_V^R(\mu^2{=}m_{\rm N}^2)$. Fig.2 shows $\frac{\alpha}{2\pi}\mathcal{G}_1^{rad}$ for the inverse $\beta$-decay reaction for a range of $E_\nu$ corresponding to

reactor-generated anti-neutrinos ($E_\nu \lesssim 8$ MeV). Here again, the pair of RC curves represents an error-band reflecting the uncertainties in $\tilde{e}_V^R(\mu^2{=}m_{\rm N}^2)$.

Fig.1 also shows $\frac{1}{m_N}\mathcal{K}_1^{recoil}$ for $\beta$-decay, while Fig.2 includes $\frac{1}{m_N}\mathcal{G}_1^{recoil}$ for the inverse $\beta$-decay reaction. It is seen that $\mathcal{K}_1^{recoil}$ is much smaller than $\mathcal{G}_1^{recoil}$, which is not unexpected, because $E_\nu$ for reactor neutrinos is significantly higher than the anti-neutrino energy involved in neutron $\beta$-decay ($E_\nu {\lesssim}\, 0.78$ MeV). We point out, however, that the difference between $\mathcal{K}_1^{recoil}$ and $\mathcal{G}_1^{recoil}$ is further enhanced by the fact that, as seen in Eqs.(17) and (18), there is near cancellation among the various terms contributing to $\mathcal{K}_1^{recoil}$, whereas they add up for $\mathcal{G}_1^{recoil}$.

It may be of interest to consider the "total recoil" correction, which represents the combined effect of the $m_N^{-1}$ terms in the lagrangian, Eq.(6), and the $m_N^{-1}$ correction in the phase space factor, Eq.(9). The differential decay rate, Eq.(7), after angle-integration can be rewritten as

$$\frac{{\rm d}\Gamma_\beta}{{\rm d}E} = \frac{{\rm d}\Gamma_\beta^{(0)}}{{\rm d}E}\left\{1+\frac{\alpha}{2\pi}\mathcal{K}_1^{rad}(\beta)+\frac{1}{m_N}\mathcal{K}_1^{\rm total-recoil}(\beta)\right\}\,, \tag{20}$$

where ${\rm d}\Gamma_\beta^{(0)}/{\rm d}E = (G_F V_{ud})^2\Big[F(Z=1,E)E^2\beta(E^{max}-E)^2/(2\pi^3)\Big](1+3g_A^2)$ is the standard LO result, and the total recoil correction, $\mathcal{K}_1^{\rm total-recoil}$ is given by

$$\mathcal{K}_1^{\rm total-recoil}(\beta) = \mathcal{K}_1^{recoil}(\beta) + 3E - E^{max} - \left(\frac{1-g_A^2}{1+3g_A^2}\right)\beta^2 E + \mathcal{O}(m_N^{-1})\,. \tag{21}$$

Meanwhile, Eq.(8) (after integrated over the angle) can be rewritten as

$$\sigma_{inv-\beta} = \sigma_{inv-\beta}^{(0)}\left\{1+\frac{\alpha}{2\pi}\mathcal{G}_1^{rad}(\beta)+\frac{1}{m_N}\mathcal{G}_1^{\rm total-recoil}(\beta)\right\}\,, \tag{22}$$

where $\sigma_{inv-\beta}^{(0)} = (G_F V_{ud})^2(\tilde{E}^2\tilde{\beta}/\pi)(1+3g_A^2)$ is the standard LO result, and the "total" recoil correction, $\mathcal{G}_1^{\rm total-recoil}(\beta)$, is given by[4]

$$\mathcal{G}_1^{\rm total-recoil}(\beta) = \mathcal{G}_1^{recoil}(\beta) - \tilde{E} - \left(\frac{1+\tilde{\beta}^2}{\tilde{\beta}^2}\right)\left(E_\nu + \frac{\Delta_N^2 - m_e^2}{2\tilde{E}}\right) + \left(\frac{1-g_A^2}{1+3g_A^2}\right)E_\nu + \mathcal{O}(m_N^{-1})\,, \tag{23}$$

[4] The last term in Eq.(20) in Ref. [12] involved an error in its evaluation. In Eq.(23) in the present article, this error has been corrected.

where $\tilde{E} = E_\nu - (m_n - m_p) = E_\nu - \Delta_N$ is the "lowest-order positron energy" (*i.e.,* the energy corresponding to a case where the neutron recoil is neglected), and $\tilde{\beta}$ is the corresponding velocity: $\tilde{\beta} = \sqrt{\tilde{E}^2 - m_e^2}/\tilde{E}$.[5]

Fig.1 displays $\mathcal{K}_1^{\rm total-recoil}$, while Fig.2 shows $\mathcal{G}_1^{\rm total-recoil}$. We note that $|\mathcal{K}_1^{\rm total-recoil}|$ is significantly smaller than $|\mathcal{G}_1^{\rm total-recoil}|$. As for comparison between the RCs and the recoil corrections, Fig.1 and Fig.2 indicate that the recoil corrections are much smaller than the RCs for neutron $\beta$-decay. By contrast, for inverse $\beta$-decay process, $\mathcal{G}_1^{rad}$ and $\mathcal{G}_1^{\rm total-recoil}$ are of comparable size and have opposite signs. It is also to be noted that, as seen in Fig. 2, for the inverse $\beta$-decay reaction the recoil corrections and the RCs have very distinct energy dependencies. Thus both corrections must be carefully considered in analyzing high-precision data used to deduce $\theta_{13}$.

To summarize, we have presented a brief discussion of numerical consequences of the radiative corrections (RCs) and the recoil corrections for neutron $\beta$-decay, Eq.(2), and inverse $\beta$-decay, Eq.(1), calculated up to next-to-leading order (NLO) in HB$\chi$PT. The numerical results reported here are obtained with the use of the value of the LEC, $\tilde{e}_V^R(\mu^2)$, that has been deduced from comparison of our HB$\chi$PT results with those obtained in the S-M approach. This hybrid nature of our analysis should eventually be replaced by a more rigorous determination of $\tilde{e}_V^R(\mu^2)$ through a direct fit to the neutron $\beta$-decay data. It is expected, however, that the basic features of the RCs are already visible in the present hybrid treatment. The recoil corrections, which up to NLO do not involve the LEC, are not affected by the hybrid nature of our numerical calculation.

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# References

[1] Y. Abe *et al.* (Double Chooz Collaboration), Phys. Rev. Lett., **108**, 131801 (2012); arXiv:1112.6353.

[2] F.P. An *et al.* (Daya Bay Collaboration), arXiv:1203.1669;

[5] Here, $E$ and $\beta$ in Eqs. (8) and (10) have been expanded in powers of $m_{\rm N}^{-1}$, see Ref. [12]. It is to be noted that, although the $1/m_N$-expanded expression in in Eq. (23) diverges at the threshold, $\sigma_{inv-\beta}$ vanishes at the threshold when the correct phase space factor, $f(E)$ of Eq. (10), is used.

[3] J.K. Ahn *et al.* (RENO Collaboration), arXiv:1204.0626 [hep-ex]

[4] P. Adamson *et al.* (MINOS Collaboration), Phys. Rev. D **82**, 051102 (2010); Phys. Rev. Lett. **106**, 181801 (2011); Phys. Rev. Lett. **107**, 021801 (2011).

[5] M. Fukugita and T. Kubota, Acta Phys. Polon. **B35**, 1687 (2004)[arXiv:hep-ph/0403149]; Erratum: arXiv:hep-ph/0403149

[6] P. Vogel, Phys. Rev. D29, 1918 (1984)

[7] S.A. Fayans, Yad. Fiz. **42**, 929 (1985) [Sov. J. Nucl. Phys. **42**, 590 (1985)].

[8] A. Sirlin, Nucl. Phys. B**71**, 29 (1974); A. Sirlin, Nucl. Phys. B**100**, 291 (1975); A. Sirlin, arXiv:hep-ph/0309187 (2003)

[9] W.J. Marciano and A. Sirlin, Phys. Rev. Lett. **56**, 22 (1986).

[10] P. Vogel and J.F. Beacom, Phys. Rev. D **60**, 053003 (1999).

[11] S. Ando *et al.*, Phys. Lett. B **595**, 250 (2004).

[12] U. Raha, F. Myhrer and K. Kubodera, Phys. Rev. C **85**, 045502 (2012); arXiv:1112.2007[hep-ph].

[13] A. Sirlin, Phys. Rev. D bf 84, 014021 (2011); arXiv:1105.2842 [hep-ph].

[14] V. Bernard, N. Kaiser and U.-G. Meißner, Int. J. Mod. Phys. **E4**, 193 (1995).

[15] S. Scherer, Adv. Nucl. Phys., **27**, 2001 (2003).

[16] V. Bernard, Prog. Part. Nucl. Phys. **60**, 82 (2008).

[17] *Review of Particle Physics* by the Particle Data Group, C. Amsler *et al.*, Phys. Lett. **B 667**, 1 (2008)

[18] W.J. Marciano and A. Sirlin, Phys. Rev. Lett. **96**, 032002 (2006).

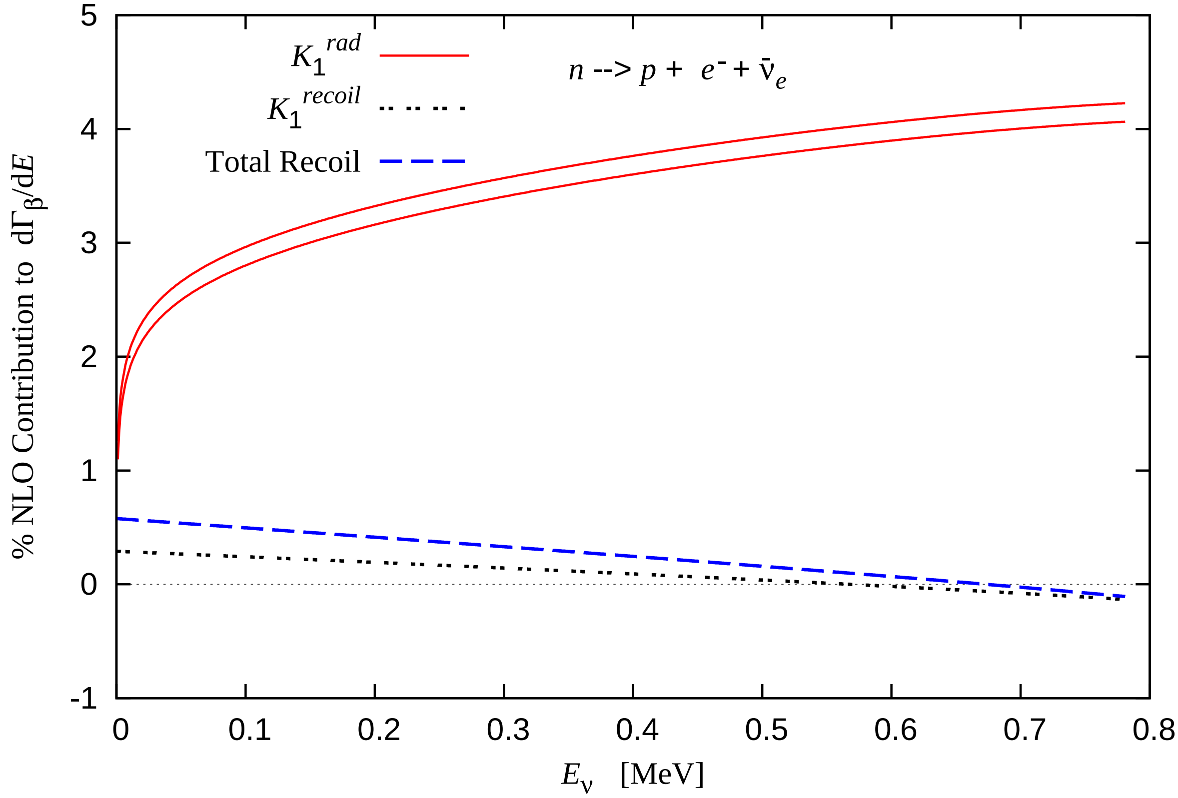


Figure 1: For neutron $\beta$-decay, the radiative correction, $\frac{\alpha}{2\pi}\mathcal{K}_1^{rad}$ in Eq. (13), and the recoil $m_N^{-1}$ correction, $m_N^{-1}\,\mathcal{K}_1^{recoil}$ in Eq. (17), are plotted in % as functions of the emitted anti-neutrino energy $E_\nu$. The dashed curve labeled "Total Recoil" gives $\frac{1}{m_N}\mathcal{K}_1^{\text{total}-\text{recoil}}$ in Eq.(21). The pair of radiative correction curves gives an "error band" that reflects uncertainties in the values of the LEC, $\tilde{e}_V^R(m_{\rm N}^2)$, see the text.

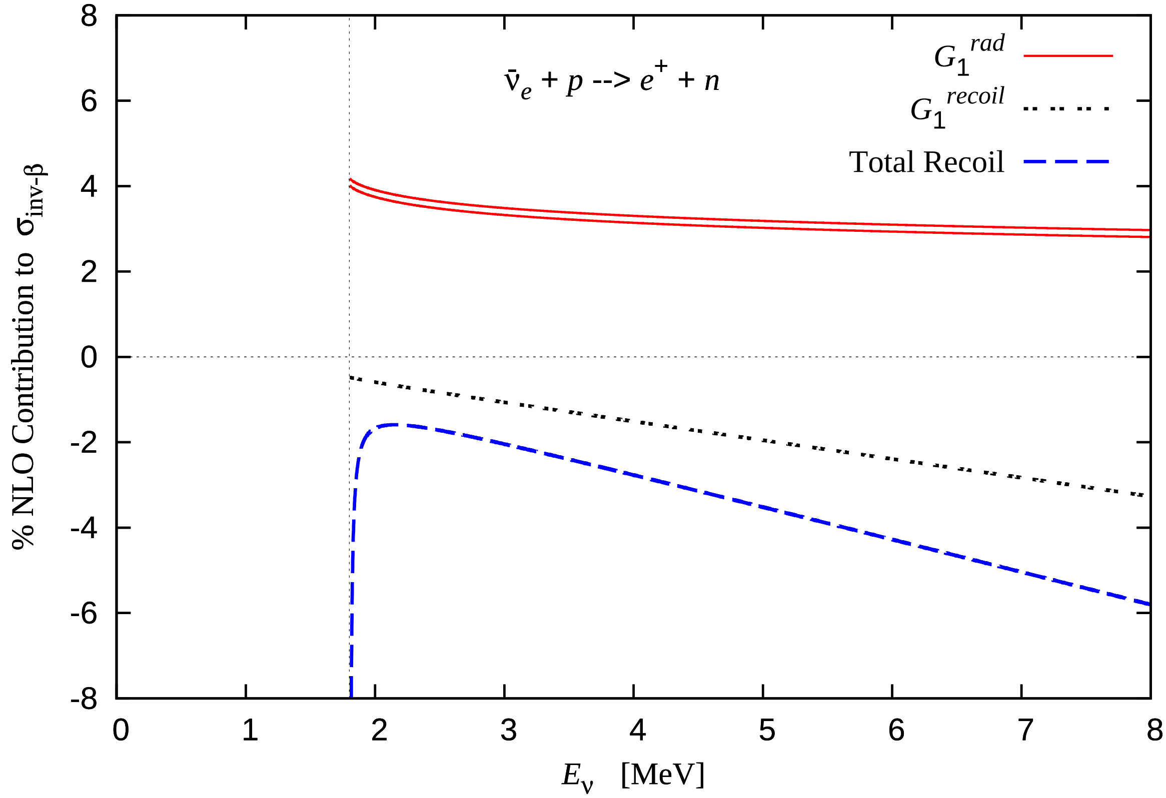


Figure 2: For the the reaction $\bar{\nu}_e + p \to e^+ + n$, the radiative correction, $\frac{\alpha}{2\pi}\,\mathcal{G}_1^{rad}$ in Eq. (13), and the recoil correction, $m_N^{-1}\,\mathcal{G}_1^{recoil}$ in Eq. (18), are plotted (in per cent) as functions of the incoming anti-neutrino energy $E_\nu$. The dashed curve labeled "Total Recoil" gives $\frac{1}{m_N}\mathcal{G}_1^{\text{total}-\text{recoil}}$ in Eq.(23). The pair of radiative correction curves gives an "error band" that reflects uncertainties in the values of the LEC, $\tilde{e}_V^R(m_{\rm N}^2)$, see the text. The vertical dotted line indicates the threshold ($E_\nu \simeq 1.8$ MeV).